JOURNAL OF MATERIALS SCIENCE 26 (1991) 3392–3395

Fracture mechanics and cavitation in rubber-like solids

A. N. GENT, C. WANG
Institute of Polymer Engineering, The University of Akron, Akron, OH 44325-0301, USA

Conditions for propagation of a pressurized crack within a rubber-like solid are derived in terms of the elastic properties of rubber, the fracture energy G_c and the initial radius r_0 of the crack. A previously proposed criterion, that the critical internal pressure P_c for crack growth is given by $5E/6$, where E is the tensile (Young) modulus of elasticity, is shown to be inadequate both for small cracks, when the stiffening of rubber at high strains must be taken into account, and for large cracks, when the critical degree of inflation is so small that the assumptions leading to $P_c = 5E/6$ do not apply. However, this simple criterion is found to remain a useful guide for cracks having initial radii lying in an intermediate range, such that $r_0 E/G_c$ lies between about 0.0005 and 1. For representative rubber-like solids, this corresponds to the range $r_0 = 0.5\,\mu$m to 1 mm.

1. Introduction

If a spherical cavity in a rubber-like solid is subjected to internal pressure it will expand in a highly non-linear way. For a material obeying the simple kinetic theory of rubber elasticity, the relation between inflating pressure P and expansion ratio λ of the cavity radius (Fig. 1a) is

$$P/E = (5 - 4\lambda^{-1} - \lambda^{-4})/6 \qquad (1)$$

where E is Young's modulus of elasticity of the rubber at small tensile strains [1, 2]. This relation is also a reasonably good guide for materials showing somewhat more complex elastic behaviour [2]. It predicts that, at a critical inflation pressure P_c, given by $5E/6$, the cavity will expand without limit. In practice, it will tear open to form an internal crack when the maximum extensibility of the rubber is reached.

We now examine the hypothesis that rubber-like solids contain small cavities that develop into internal cracks as a result of elastic expansion. Note that Equation 1 does not contain the initial radius of the cavity, so that the actual size is not important at this point. Several observations suggest that this concept is valid.

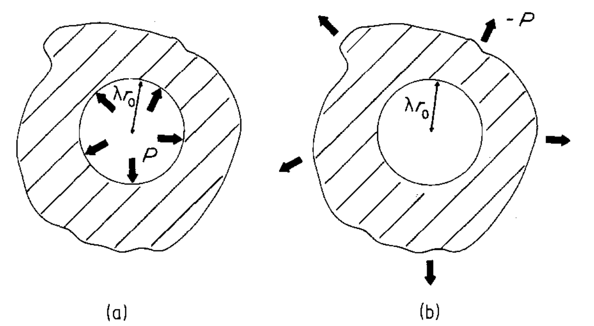

Figure 1 (a) Sketch of an inflated spherical void; (b) spherical void in a medium under a far-field triaxial tension (negative pressure) of $-P$.

When rubber blocks are supersaturated with high-pressure dissolved gases or liquids and the supersaturation pressure exceeds $5E/6$, then visible bubbles appear in the interior [3, 4]. Each bubble corresponds to an internal crack. Whenever a local dilatant stress (triaxial tension or negative hydrostatic pressure) of $5E/6$ is set up in rubber, then large internal cracks appear [2, 5–7]. (For incompressible materials, like rubber, a dilatant stress acting at a distance is equivalent to a pressure of the same magnitude acting within a cavity, Fig. 1). Thus, there is a considerable body of evidence that rubber-like solids contain small cavities, and that these cavities develop into large internal cracks when a critical pressure or dilatant stress of $5E/6$ is set up.

However, when the volume of material under dilatant stress is small, the critical value seems to be larger than $5E/6$ [5, 6]. And, in general, when a larger stress than this is applied, more cracks appear [3]. These observations suggest that rubber-like solids contain cavities with a wide range of sizes, the smaller ones requiring a larger pressure or stress to grow into visible bubbles or cracks. Only the largest ones will become cracks at a stress of $5E/6$.

One possible reason that higher stresses are necessary to cause growth of smaller voids is that surface energy provides an additional restraint upon expansion; a factor which becomes more important for small voids with relatively high surface areas. An extra term appears on the right-hand side of Equation 1, $2\gamma/Er$, where γ is the surface energy of rubber and r is the void radius [3, 8]. However, attention is focused here on another restraint on cavitation: the need for a critical amount of mechanical energy to be released by propagation of a tear.

According to Griffith's fracture criterion [9], no tearing will take place unless sufficient energy is released to meet the requirements for fracture. As shown

0022–2461/91 $03.00 + .12

below, when this criterion is applied to a simple model of a precursor void it leads to the conclusion that voids smaller than a certain size will not release enough strain energy to grow by tearing, even when the "critical" stress, $5E/6$, is attained, and therefore they will not develop into large internal cracks at this stress. In fact, we conclude that they will not tear open at any level of strain, however high, provided that Equation 1 applies.

In view of the fact that internal cracks do appear at sufficiently high stresses, it is necessary to modify the elastic solution, Equation 1, to take into account the departure of rubber-like materials from the simple kinetic theory of rubber-like elasticity at large deformations. A number of other elastic relations for expansion of a spherical cavity have been considered by Chou-Wang and Horgan [10], but they did not employ Griffith's criterion for fracture. A simple empirical relation for the elastic behaviour of rubber at large deformations is used here to derive the elastic energy released by growth of a cavity, and hence the conditions for formation of internal cracks by the expansion of small voids.

Griffith's fracture criterion was applied to this problem several years ago [11]. The results obtained in the first part of this paper are similar, with small but significant differences that are attributed to a different method of evaluating the rate of release of strain energy as the initial crack grows. Thus, the first part of the present discussion corroborates this earlier study, at least in a semi-quantitative way. It leads to a reconsideration of the elastic behaviour at high strains, as discussed in the second part, in order to understand the mechanism of growth of very small cracks.

2. Energy requirements for void growth

We first express Equation 1 in terms of the volume V and V_o of the void in the inflated and uninflated state, where

$$\lambda^3 = V/V_o \quad (2)$$

Strain energy W stored in the material is given by

$$W = \int_{V_o}^{V} P \, dV \quad (3)$$

This result must now be expressed in terms of the area, A, of a hypothetical crack, so that the rate $\partial W/\partial A$ of release of energy as the crack grows can be evaluated. Griffith's fracture criterion then takes the form

$$-(\partial W/\partial A)|_V \geq G_c \quad (4)$$

where G_c is the fracture energy of the material per unit area torn through. In order to carry out this calculation, we assume that the initial void consisted of a planar circular crack of radius r_o, which became inflated into an initial spherical void of the same radius r_o under a negligibly small initial pressure or stress, before further expansion to a volume V under pressure P. The initial void volume V_o is thus given by

$$V_o = 4\pi r_o^3/3 \quad (5)$$

and the crack area A is

$$A = \pi r_o^2 \quad (6)$$

Griffith's fracture criterion, Equation 4, then becomes

$$\lambda^4(\partial/\partial\lambda)\left[\lambda^{-3}\int_1^{\lambda}(P/E)\,\lambda^2\,d\lambda\right] \geq G_c/2r_oE \quad (7)$$

Denoting the left-hand side of Equation 7 by $F(\lambda)$, the condition for growth of the initial crack is then

$$F(\lambda) \geq G_c/2r_oE \quad (8)$$

For a material obeying the pressure–inflation relation given in Equation 1, $F(\lambda)$ is obtained from Equation 7 as

$$F(\lambda) = (1 + \lambda^2 - 2\lambda^{-1})/3 \quad (9)$$

This result may be compared with that obtained by Williams and Schapery [11]:

$$F(\lambda) = (2\lambda^2 + \lambda^{-4} - 3)/12 \quad (10)$$

assuming that the extension ratio λ does not change as the crack increases in size. This assumption, and hence Equation 10, is thought to be incorrect. Note also that Equation 10 predicts crack growth at a value of λ of less than unity, that is, at a sufficiently large compressive deformation, and this seems inherently unlikely. However, the two results are qualitatively similar for positive pressures and lead to the same general conclusions. Equations 7 and 9 are employed hereafter.

These equations give a necessary condition for growth of an initial crack by tearing, which however is unlikely to be met for small cracks. For example, on putting $G_c = 100\ \mathrm{J\,m^{-2}}$, a rather small value for rubber, and $E = 2$ MPa, a typical value, Equations 8 and 9 are satisfied only for cracks having a radius of 1 μm or larger, even when the maximum elastic expansion ratio of the cavity (before tearing) is given the rather large value of 10 ×. Thus, cavities of much smaller size than this, say of the order of 0.1 μm in radius, could not be made to tear open by internal pressure or external triaxial stress according to Equations 8 and 9, for any reasonable value of the elastic expansion ratio λ.

The reason for this anomaly is clear. Equation 1, upon which Equation 9 is based, was derived for particularly simple elastic materials that obey the kinetic theory of rubber-like elasticity. But real materials cease to follow this theory at high strains, near the point of rupture. Instead, they develop much higher stresses than predicted. In order to treat fractures initiated by small cavities, it is thus necessary to modify Equation 1 to take these departures at high extension ratios into account. This is attempted in an approximate way in the following section.

3. Energy release rates at large expansion ratios

Equation 1 is based on the simple kinetic-theory relation between stress t and extension ratio λ for rubber in equi-biaxial extension [12]:

$$t/E = (\lambda^2 - \lambda^{-4})/3 \quad (11)$$

We now consider the effect of an additional (empirical) term $\Delta t/E$ on the right-hand side of Equation 11 that describes, at least to a first approximation, the increasing stiffness of rubber at large strains. Such an additional stress must satisfy the following conditions:

(i) $\Delta t = 0$ when $\lambda = 1$
(ii) $\Delta t/E(\lambda - 1) = 0$ when $\lambda = 1$
(iii) Δt becomes extremely large when $\lambda = \lambda_m$, where λ_m is a limiting extension ratio.

A simple relation consistent with these requirements is

$$\Delta t = k(\lambda - 1)^2/(\lambda_m - 1)(\lambda_m - \lambda) \qquad (12)$$

where k is a new elastic constant describing the behaviour at large strains. For simplicity, k is assumed later to be equal to E, although both the form of the relation for additional stress and the magnitude of the corresponding elastic constant are, in principle, obtainable by experiment.

Equation 12 leads to an extra term on the right-hand side of Equation 1 for inflation pressure:

$$\begin{aligned}\Delta P/4k(\lambda_m - 1) = {} & A\ln[(\lambda_m - 1)/(\lambda_m - \lambda)] \\ & + B\ln[(\lambda^2 + \lambda + 1)/3] \\ & + (2/3^{1/2})(C - B) \\ & [\tan^{-1}\{(2\lambda + 1)/3^{1/2}\} \\ & - \tan^{-1}3^{1/2}] + D\ln\lambda \qquad (13)\end{aligned}$$

where

$$\begin{aligned} A &= (\lambda_m - 1)/[\lambda_m(\lambda_m^2 + \lambda_m + 1)] \\ B &= (\lambda_m + 2)/2(\lambda_m^2 + \lambda_m + 1) \\ C &= (2\lambda_m + 1)/(\lambda_m^2 + \lambda_m + 1) \\ D &= -1/\lambda_m \end{aligned}$$

This relation for ΔP leads in turn to an extra term $\Delta F(\lambda)$ on the right-hand side of Equation 9 for $F(\lambda)$, given by

$$\begin{aligned}\Delta F(\lambda)/4(\lambda_m - 1) = {} & -(\lambda - 1) + A\lambda_m^3\ln \\ & [(\lambda_m - 1)/(\lambda_m - \lambda)] \\ & - B\ln 3 + (B/3)\ln(\lambda^6 + 3\lambda^5 \\ & + 6\lambda^4 + 7\lambda^3 + 6\lambda^2 + 3\lambda \\ & + 1) + (\lambda_m^2 + \lambda_m + 1)^{-1} \\ & \{3^{1/2}\lambda_m\tan^{-1}[(2\lambda + 1)/ \\ & 3^{1/2}] - (\pi/3^{1/2})\lambda_m\} \qquad (14)\end{aligned}$$

The predictions of Equations 1, 9, 13 and 14 for the critical extension ratio λ_c and corresponding pressure P_c are shown in Figs 2 and 3 as functions of reduced crack size: r_0E/G_c. For these illustrative calculations the elastic constant k has been given the value E and the limiting extension ratio λ_m has been assumed to be 10.

When the crack radius is small, less than $10^{-4}G_c/E$, then the calculated extension ratio λ_c is close to the maximum possible value λ_m, Fig. 2. Under these circumstances the present empirical stress–strain relation, Equation 12, is unlikely to be valid and the values deduced for P_c, shown in Fig. 3, will be quite inexact. On the other hand, when the crack radius is greater than about G_c/E, then the calculated extension ratio is quite small, less than ~ 1.2. Neglect of the pressure and strain energy involved in inflating the initial crack into a spherical cavity is then obviously incorrect. We expect, therefore, that the present analysis will hold only over an intermediate range: $10^{-4} \leqslant r_0E/G_c \leqslant 1$.

Results obtained from Equations 1, 9, 13 and 14 are shown as a full curve in Fig. 3. The asymptotic pressure of $5E/6$ from Equation 1 is represented by the horizontal broken line. It is clear that the additional empirical stress function has permitted higher critical pressures than this to be reached for a wide range of crack sizes. But, even when extensions close to the maximum possible value λ_m are required to cause crack growth, when the initial radius of the crack is relatively small, the predicted critical pressure is not particularly high, only about three times that predicted by Equation 1. We conclude that the critical pressure for crack growth is of order E for a wide range of crack sizes, from about 1×10^{-4} G_c/E to about G_c/E.

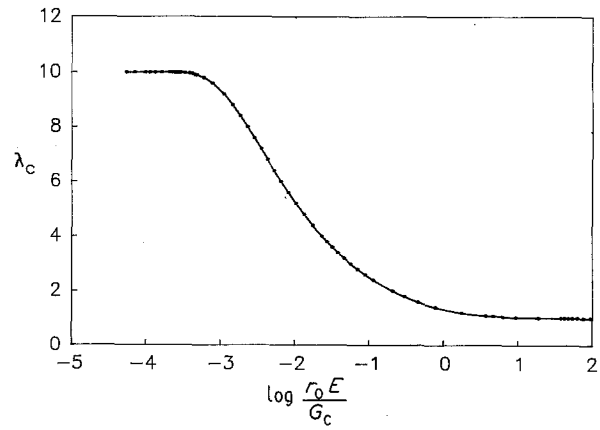

Figure 2 Calculated relation between the critical extension ratio λ_c for tearing open at the surface of a spherical cavity and the initial radius r_0 of the crack.

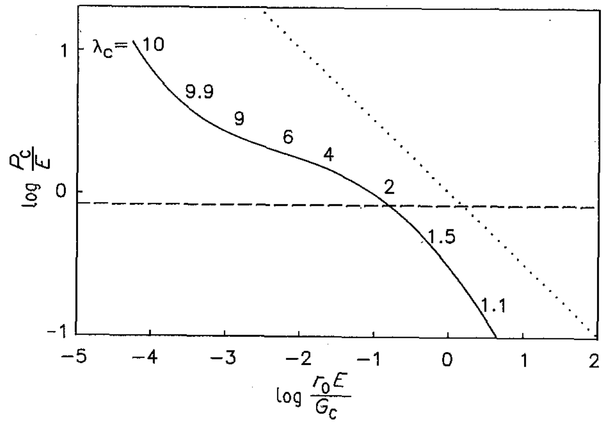

Figure 3 Calculated relations between critical pressure P_c and radius r_0 of a circular crack, scaled with respect to the fracture energy G_c and modulus E of elasticity in tension. The full line is from Equations 1, 9, 13 and 14. Corresponding values of extension ratio λ_c at the surface of the cavity are given. The dotted line is from Equation 15, valid for small extensions. The horizontal broken line denotes $P_c = 5E/6$.

For small deformations, assuming linear elasticity, the critical pressure is given by [13]

$$P_c/E = (3r_0E/\pi G_c)^{-1/2} \qquad (15)$$

This relation, represented by the dotted curve in Fig. 3, will hold for cracks of large initial radius, $r_0E/G_c > 1$, when the deformations required for crack propagation are relatively small and the crack is never inflated into a spherical void.

4. Conclusions

An energy criterion has been applied to find the conditions for propagation of a pressurized crack in a highly elastic material. The critical internal pressure P_c is found to depend strongly upon the initial radius of the crack, as pointed out earlier by Williams and Schapery [11]. This conclusion is different from a previous one, based upon the concept of a critical (large) deformation for fracture, that sufficiently large cracks will tear open at a pressure of $5E/6$, independent of crack size (2–7).

However, the new analysis, also based on the kinetic theory of rubber-like elasticity, does not account for the tearing open of small cracks. Instead, it predicts that the required strains will become unreasonably large. This difficulty has been overcome by employing an empirical stress–strain relation, consistent with the kinetic theory of rubber elasticity at low and moderate strains, but giving extremely high stresses when a limiting strain is approached. In this way, values of P_c have been calculated for a wide range of crack sizes, using an energy criterion for fracture. They are found to lie in the relatively narrow range, $3E$ to E, for crack radii ranging from $5 \times 10^{-4}G_c/E$ to G_c/E. For conventional rubbery materials, this corresponds to radii ranging from about 0.5 μm to about 1 mm. Below this range, the detailed form of the stress–strain relation at high stresses becomes important: no general conclusion can be drawn. And for larger cracks than this the simple Griffith result, Equation 15, applies.

Acknowledgements

This work forms part of a study of failure mechanisms in rubber, supported by the Office of Naval Research (Contract N00014-85-K-0222; Project Officer Dr R. S. Miller) and by grants-in-aid from Lord Corporation and 3M. The authors are also indebted to Professor A. G. Thomas for helpful discussions.

References

1. A. E. GREEN and W. ZERNA, "Theoretical Elasticity" (Oxford University Press, London, 1954) Section 3.10.
2. A. N. GENT and P. B. LINDLEY, *Proc. R. Soc.* (Lond.) **A249** (1958) 195.
3. A. N. GENT and D. A. TOMPKINS, *J. Appl. Phys.* **40** (1969) 2520.
4. R. L. DENECOUR and A. N. GENT, *J. Polym. Sci.* **6** (1968) 1853.
5. A. N. GENT and B. PARK, *J. Mater. Sci.* **19** (1984) 1947.
6. K. CHO and A. N. GENT, *ibid.* **23** (1988) 141.
7. A. E. OBERTH and R. S. BRUENNER, *Trans. Soc. Rheol.* **9** (1965) 165.
8. A. N. GENT and D. A. TOMPKINS, *J. Polym. Sci.* **7** (1969) 1483.
9. A. A. GRIFFITH, *Phil. Trans. R. Soc.* (Lond.) **A221** (1921) 163.
10. M.-S. CHOU-WANG and C. O. HORGAN, *Int. J. Solids Struct.* **25** (1989) 1239.
11. M. L. WILLIAMS and R. A. SCHAPERY, *Internat. J. Fract. Mech.* **1** (1965) 64.
12. L. R. G. TRELOAR, "Physics of Rubber Elasticity", 2nd edn (Clarendon, Oxford, 1958) p. 87.
13. R. A. SACK, *Proc. Phys. Soc. (Lond.)* **58** (1946) 729.

Received 27 March
and accepted 9 April 1990